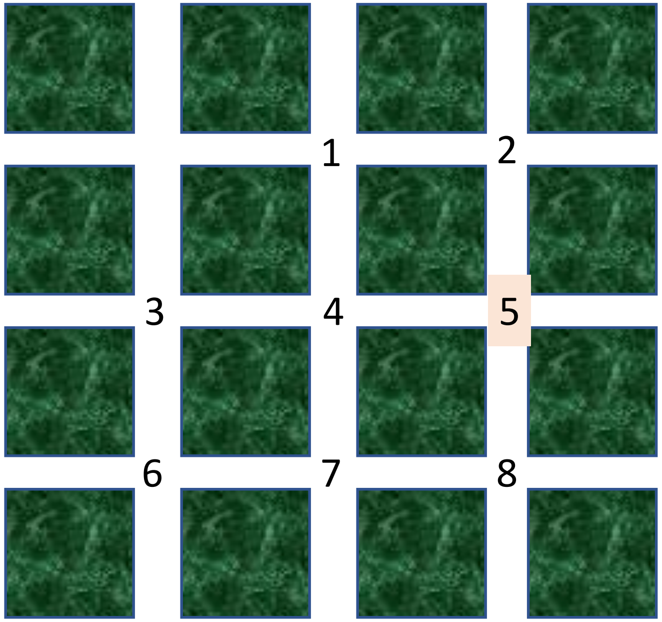
Primer: Markov chains

Lecture 11.2

Traffic officer assignment

- One traffic officer is assigned to control intersections: 1 - 8
- He is instructed to remain at a given intersection for an hour, and then either to remain at the same intersection, or move to a neighboring intersection
- To avoid establishing a pattern, he is instructed to choose his new intersection at a random basis, with each possible move being equally likely
- For example, if he starts at intersection 5, his next intersection could be 2,4,5,8 – all with equal probability ¼
- Every day he starts at the location where he stopped before



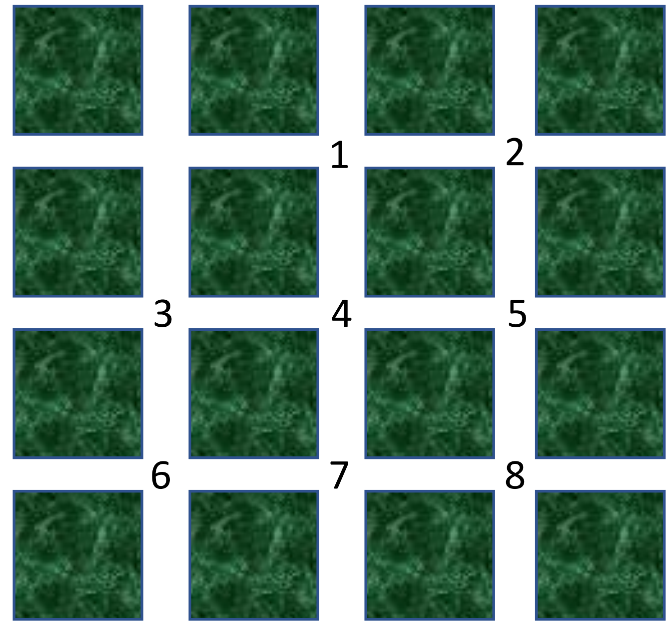
Markov chains

- The system can be in a finite number of states
- The transition from state to state is not predetermined, but rather can be specified in terms of *probabilities* which depend on the previous history of the system. This system is called a *stochastic* system
- If the transition probabilities depend only on the immediate history of the system – for example the state at current observation depends only on the state in the preceding observation – then *the process of transitions from state to state* is called a ***Markov process*** or a ***Markov chain***

Markov model for traffic officer

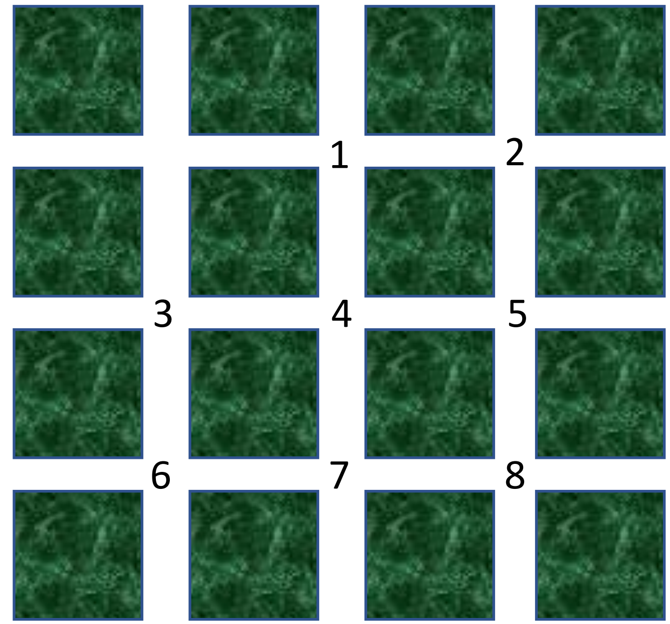
Transition probabilities

		Old intersection							
New intersection		1	2	3	4	5	6	7	8
	1	1/3	1/3	0	1/5	0	0	0	0
	2	1/3	1/3	0	0	1/4	0	0	0
	3	0	0	1/3	1/5	0	1/3	0	0
	4	1/3	0	1/3	1/5	1/4	0	1/4	0
	5	0	1/3	0	1/5	1/4	0	0	1/3
	6	0	0	1/3	0	0	1/3	1/4	0
	7	0	0	0	1/5	0	1/3	1/4	1/3
	8	0	0	0	0	1/4	0	1/4	1/3



Markov model: definitions

- *States*: intersections 1 – 8
- *Transition probability*: probability of moving from a given state to another state
- *Transition matrix*: any matrix with non-negative entries where every column sums up to 1

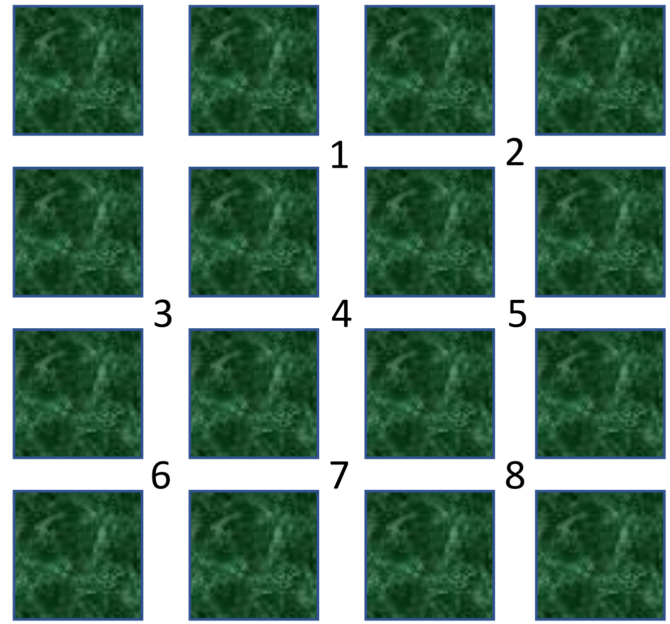


		Old intersection							
New intersection		1	2	3	4	5	6	7	8
	1	1/3	1/3	0	1/5	0	0	0	0
	2	1/3	1/3	0	0	1/4	0	0	0
	3	0	0	1/3	1/5	0	1/3	0	0
	4	1/3	0	1/3	1/5	1/4	0	1/4	0
	5	0	1/3	0	1/5	1/4	0	0	1/3
	6	0	0	1/3	0	0	1/3	1/4	0
	7	0	0	0	1/5	0	1/3	1/4	1/3
	8	0	0	0	0	1/4	0	1/4	1/3

Probability vector for state 8

Markov model: theorem

If *P* is the transition matrix of a Markov process, and *x(t)* is a probability vector for state in time *t*, then the probability vector *x(t+1) = P*x(t)*



	Old intersection								
New intersection		1	2	3	4	5	6	7	8
	1	1/3	1/3	0	1/5	0	0	0	0
	2	1/3	1/3	0	0	1/4	0	0	0
	3	0	0	1/3	1/5	0	1/3	0	0
	4	1/3	0	1/3	1/5	1/4	0	1/4	0
	5	0	1/3	0	1/5	1/4	0	0	1/3
	6	0	0	1/3	0	0	1/3	1/4	0
	7	0	0	0	1/5	0	1/3	1/4	1/3
	8	0	0	0	0	1/4	0	1/4	1/3

$x(1) = P*x(0)$

$x(2) = P*x(1) = P^2*x(0)$

$x(3) = P*x(2) = P^3*x(0)$

…

$x(n) = P*x(n-1) = P^n*x(0)$

The probabilities converge with time

- If the officer begins at intersection5, his probable locations hour-by-hour are given in the table
- The probability vectors approach a fixed vector as t increases
- All the values after 22 hours will be the same up to 3 decimal places

t	0	1	2	3	4	5	10	15	20	22
1	0	.000	.133	.116	.130	.123	.113	.109	.108	.107
2	0	.250	.146	.163	.140	.138	.115	.109	.108	.107
3	0	.000	.005	.039	.067	.073	.100	.106	.107	.107
4	0	.250	.113	.187	.162	.178	.178	.179	.179	.179
5	1	.250	.279	.190	.190	.168	.149	.144	.143	.143
6	0	.000	.000	.050	.056	.074	.099	.105	.107	.107
7	0	.000	.133	.104	.131	.125	.138	.141	.143	.143
8	0	.250	.146	.152	.124	.121	.108	.107	.107	.107

Check *markov.py*

Exercise: Migrants

- A country is divided into 3 demographic regions.
- It is determined that each year:
  - Of the residents of region 1:
    - 5% move to region 2 and 5% move to region 3
  - Of the residents of region 2:
    - 15% move to region 1 and 10% move to region 3
  - Of the residents of region 3:
    - 10% move to region 1 and 5% move to region 2

- What percentage of the country resides in each region after a long period of time?